"On the Computation of the Harmonic Components of a Series representing a Phenomenon recurring in Daily and Yearly Periods." By Lieut.-General R. STRACHEY, R.E., F.R.S. Received April 15,—Read May 13, 1886.

1.

Following the notation commonly used, the general expression for the harmonic components of the successive terms of a series representing a periodically recurring phenomenon, observed at equal intervals of time, is—

$$a_n = p_0 + p_1 \cos nz + q_1 \sin nz + p_2 \cos 2nz + q_2 \sin 2nz + \&c.,$$

where a_n is the observed value of any term in question;
z is the angular equivalent of the time interval between the observations;
n is the number of intervals from the commencement of the period to the time when the term a_n occurs;
p_0 is the mean value of all the terms for the whole period.

Then if A_n represents the sum of the terms in the above series which involve nz,
B_n the sum of the terms involving $2nz$,
C_n „ „ $3nz$, &c.,

$$a_n = p_0 + A_n + B_n + C_n + \&c.$$

Computation for a Daily Period.

2.

For a daily period of 24 hourly intervals $z = 15°$, and consequently,

$$A_n = A_{n+12};$$

$$B_n = B_{n+12} = -B_{n+6} = -B_{n+18};$$

$$C_n = C_{n+8} = C_{n+16} = -C_{n+4} = -C_{n+12} = -C_{n+20};$$

$$D_n = D_{n+6} = D_{n+12} = D_{n+18} = -D_{n+3} = -D_{n+9} = -D_{n+15} = D_{n+21};$$

whence, disregarding the terms involving multiples of z greater than $4nz$,

$$d_n = a_n - a_{n+12} = 2(A_n + C_n),$$

and

$$o_n = d_n - d_{n+4} + d_{n+8} = 2(A_n - A_{n+4} + A_{n+8}) + 2(C_n - C_{n+4} + C_{n+8});$$

but

$$A_n = p_1 \cos (n \,.\, 15^\circ) + q_1 \sin (n \,.\, 15^\circ),$$

$$A_{n+4} = p_1 \cos (n \,.\, 15^\circ + 60^\circ) + q_1 \sin (n \,.\, 15^\circ + 60^\circ),$$

$$A_{n+8} = p_1 \cos (n \,.\, 15^\circ + 120^\circ) + q_1 \sin (n \,.\, 15^\circ + 120^\circ),$$

and therefore

$$A_n - A_{n+4} + A_{n+8} = 0.$$

Wherefore

$$\theta_n = 2(C_n - C_{n+4} + C_{n+8}) = 6C_n ; \qquad \text{and} \qquad C_n = \tfrac{1}{6}\theta_n,$$

and

$$A_n = \tfrac{1}{2}d_n - \tfrac{1}{6}\theta_n.$$

In like manner,

$$S_n = a_n + a_{n+12} = 2(p_0 + B_n + D_n),$$

and

$$\Sigma_n = S_n - S_{n+6} = 2(B_n - B_{n+6} + D_n - D_{n+6}) = 4B_n ; \text{ and } B_n = \tfrac{1}{4}\Sigma_n,$$

also

$$\sigma_n = S_n + S_{n+6} = 2(p_0 + B_n + B_{n+6} + D_n + D_{n+6}) = 2(p_0 + 2D_n),$$

and

$$\sigma_{n+3} = S_{n+3} + S_{n+9} = 2(p_0 - 2D_n),$$

whence

$$\psi_n = \sigma_n - \sigma_{n+3} = 8D_n ; \qquad \text{and} \qquad D_n = \tfrac{1}{8}\psi_n.$$

The successive values of A, B, C, and D, thus obtained will give with a considerable degree of accuracy, the p q coefficients, and the entire series of harmonic components of the observed quantities. It will at once be seen that—

$$A_0 = p_1 ; \; B_0 = p_2 ; \; C_0 = p_3 ; \; D_0 = p_4.$$

$$A_6 = q_1 ; \; B_3 = q_2 ; \; C_2 = q_3 ; \; D_1 + D_2 = 2 \sin 60 q_4 = \tfrac{7}{4} q_4, \text{ nearly.}$$

and

$$\left.\begin{aligned} p_1 &= \tfrac{1}{2}d_0 - \tfrac{1}{6}\theta_0 ; & q_1 &= \tfrac{1}{2}d_6 + \tfrac{1}{6}\theta_2. \\ p_2 &= \tfrac{1}{4}\Sigma_0 ; & q_2 &= \tfrac{1}{4}\Sigma_3. \\ p_3 &= \tfrac{1}{6}\theta_0 = \tfrac{1}{6}(d_0 - d_4 + d_8) ; & q_3 &= \tfrac{1}{6}\theta_2 = \tfrac{1}{6}(d_2 - d_6 + d_{10}). \\ p_4 &= \tfrac{1}{8}(\sigma_0 - \sigma_3) ; & q_4 &= \tfrac{1}{14}(\sigma_1 + \sigma_2 - \sigma_4 - \sigma_5). \end{aligned}\right\} \quad (1.)$$

3.

The equations involving the harmonic coefficients, arising from the series of observed quantities, are usually solved according to the method of least squares; and, writing Δ_1 for $(d_1 - d_{11})$, and δ_1 for $(d_1 + d_{11})$, and so on, the resulting values of p q thus obtained are as follows :—

$$\left.\begin{aligned}
p_1 &= \tfrac{1}{12}\{d_0 + \Delta_1 \sin 75 + \Delta_2 \sin 60 + \Delta_3 \sin 45 + \Delta_4 \sin 30 \\
&\qquad + \Delta_5 \sin 15\} \\
q_1 &= \tfrac{1}{12}\{d_6 + \delta_1 \sin 15 + \delta_2 \sin 30 + \delta_3 \sin 45 + \delta_4 \sin 60 \\
&\qquad + \delta_5 \sin 75\} \\
p_2 &= \tfrac{1}{12}\{\Sigma_0 + (\Sigma_1 - \Sigma_5) \sin 60 + (\Sigma_2 - \Sigma_4) \sin 30\} \\
q_2 &= \tfrac{1}{12}\{\Sigma_3 + (\Sigma_1 + \Sigma_5) \sin 30 + (\Sigma_2 + \Sigma_4) \sin 60\} \\
p_3 &= \tfrac{1}{12}\{\theta_0 + (\theta_1 - \theta_3) \sin 45\} \\
q_3 &= \tfrac{1}{12}\{\theta_2 + (\theta_1 + \theta_3) \sin 45\} \\
p_4 &= \tfrac{1}{12}\{\psi_0 + (\psi_1 - \psi_2) \sin 30\} \\
q_4 &= \tfrac{1}{12}(\psi_1 + \psi_2) \sin 60.
\end{aligned}\right\} (2.)$$

To these may be added

$$\begin{aligned}
p_5 &= \tfrac{1}{12}\{d_0 + \Delta_1 \sin 15 - \Delta_2 \sin 60 - \Delta_3 \sin 45 + \Delta_4 \sin 30 \\
&\qquad + \Delta_5 \sin 75\}, \\
q_5 &= \tfrac{1}{12}\{d_6 + \delta_1 \sin 75 + \delta_2 \sin 30 - \delta_3 \sin 45 - \delta_4 \sin 60 + \delta_5 \sin 15\}, \\
p_6 &= \tfrac{1}{12}\{\Sigma_0 + \Sigma_4 - \Sigma_2\}, \\
q_6 &= \tfrac{1}{12}\{\Sigma_1 - \Sigma_3 + \Sigma_5\}, \\
p_7 &= \tfrac{1}{12}\{d_0 - \Delta_1 \sin 15 - \Delta_2 \sin 60 + \Delta_3 \sin 45 + \Delta_4 \sin 30 \\
&\qquad - \Delta_5 \sin 75\}, \\
q_7 &= \tfrac{1}{12}\{-d_6 + \delta_1 \sin 75 - \delta_2 \sin 30 - \delta_3 \sin 45 + \delta_4 \sin 60 \\
&\qquad + \delta_5 \sin 15\}, \\
p_8 &= \tfrac{1}{12}\{(\sigma_0 + \sigma_3) - (\sigma_1 + \sigma_2 + \sigma_4 + \sigma_5) \sin 30\}, \\
q_8 &= \tfrac{1}{12}\{(\sigma_1 + \sigma_4) - (\sigma_2 + \sigma_5)\} \sin 60.
\end{aligned}$$

4.

The values of the p q coefficients may, however, be obtained otherwise, in a form which is somewhat simpler for computation, and not sensibly less accurate.

Since $$A_n = p_1 \cos n15^\circ + q_1 \sin n15^\circ,$$

$$A_0 = p_1, \qquad \text{and} \qquad A_6 = q_1,$$

and putting $[A_1, A_n]$ for the sum of the series of quantities A_1 to A_n, and [cos 75, cos 15] for the sum of cos 75+cos 60+cos 45+cos 15, it follows that

$$[A_1, A_5] = p_1[\cos 75^\circ, \cos 15^\circ] + q_1[\sin 15^\circ, \sin 75^\circ],$$

$$[A_7, A_{11}] = -p_1[\cos 15^\circ, \cos 75^\circ] + q_1[\sin 75^\circ, \sin 15^\circ],$$

and

$$A_0 + [A_1, A_5] - [A_7, A_{11}] = p_1\{1 + 2[\sin 15^\circ, \sin 75^\circ]\} = 7{\cdot}5956 p_1,$$

$$A_6 + [A_1, A_5] + [A_7, A_{11}] = q_1\{1 + 2[\sin 15^\circ, \sin 75^\circ]\} = 7{\cdot}5956 q_1.$$

In like manner,

$$B_0 + [B_1 + B_2] - [B_4 + B_6] = p_2\{1 + 2[\sin 30 + \sin 60]\} = 3{\cdot}732 p_2,$$

$$B_3 + [B_1 + B_2] + [B_4 + B_5] = q_2\{1 + 2[\sin 30 + \sin 60]\} = 3{\cdot}732 q_2,$$

$$C_0 + C_1 - C_3 = p_3\{1 + 2 \sin 45^\circ\} = 2{\cdot}4142 p_3,$$

$$C_1 + C_2 + C_3 = q_3\{1 + 2 \sin 45^\circ\} = 2{\cdot}4142 q_3,$$

$$D_0 + D_1 - D_2 = p_4\{1 + 2 \sin 30\} = 2 p_4,$$

$$D_1 + D_2 = q_4 \,.\, 2 \sin 60 = 1{\cdot}7321 q_4.$$

Substituting for A, B, C, D, their values in terms of d, Δ, δ, &c., we have—

$$7{\cdot}5956 p_1 = \tfrac{1}{2}\{d_0 + [\Delta_1, \Delta_5] + \tfrac{1}{3}(\theta_0 + \theta_1 - \theta_3)\},$$

$$7{\cdot}5956 q_1 = \tfrac{1}{2}\{d_6 + [\delta_1, \delta_6] - \tfrac{1}{3}(\theta_1 + \theta_2 + \theta_3)\},$$

$$3{\cdot}732 p_2 = \tfrac{1}{4}\{\Sigma_0 + [\Sigma_1 + \Sigma_2] - [\Sigma_4 + \Sigma_5]\},$$

$$3{\cdot}732 q_2 = \tfrac{1}{4}\{\Sigma_3 + [\Sigma_1 + \Sigma_2] + [\Sigma_4 + \Sigma]\},$$

$$2{\cdot}4142 p_3 = \tfrac{1}{6}\{\theta_0 + \theta_1 - \theta_3\} = \tfrac{1}{6}\alpha,$$

$$2{\cdot}4142 q_3 = \tfrac{1}{6}\{\theta_1 + \theta_2 + \theta_3\} = \tfrac{1}{6}\beta,$$

$$2 p_4 = \tfrac{1}{8}\{\psi_0 + \psi_1 - \psi_2\},$$

$$1{\cdot}7321 q_4 = \tfrac{1}{8}\{\psi_1 + \psi_2\}.$$

And

$$\left.\begin{aligned}
p_1 &= \cdot 06583\{d_0+[\Delta_1,\ \Delta_5]+\tfrac{1}{3}\alpha\},\\
q_1 &= \cdot 06583\{d_6+[\delta_1,\ \delta_5]-\tfrac{1}{3}\beta\},\\
p_2 &= \cdot 06699\{\Sigma_0+[\Sigma_1+\Sigma_2]-[\Sigma_4+\Sigma_5]\},\\
q_2 &= \cdot 06699\{\Sigma_3+[\Sigma_1+\Sigma_2]+[\Sigma_4+\Sigma_5]\},\\
p_3 &= \cdot 06904\{\theta_0+\theta_1-\theta_3\} = \cdot 06904\{[d_0+\Delta_1]-[\Delta_3,\ \Delta_5]\} = \alpha,\\
q_3 &= \cdot 06904\{\theta_1+\theta_2+\theta_3\} = \cdot 06904\{[\delta_1,\ \delta_3]-[\delta_5+d_6]\} = \beta,\\
p_4 &= \tfrac{1}{16}\{\psi_0+\psi_1-\psi_2\} = \cdot 0625\{\psi_0+\psi_1-\psi_2\},\\
q_4 &= \cdot 07217\{\psi_1+\psi_2\}.
\end{aligned}\right\}\ (3.)$$

The expressions before given for p_6, q_6, and p_8, q_8, are very readily computed, the multiplier for q_8, $\frac{1}{12}\sin 60 = \cdot 07217$, being the same as that for q_4.

The computation of p_5, q_5, and p_7, q_7, may be rendered somewhat easier as follows:—

$$\begin{aligned}
p_5 &= \tfrac{1}{12}(2d_0+\Delta_4)+\cdot 102(\Delta_1+\Delta_5)-p_1,\\
p_7 &= \tfrac{1}{12}(2d_0+\Delta_4)+\cdot 059(\Delta_1+2\Delta_3-\Delta_5)-p_1,\\
q_5 &= \tfrac{1}{12}(2d_6+\delta_2)+\cdot 102(\delta_1+\delta_5)-q_1,\\
q_7 &= -\tfrac{1}{12}(2d_6+\delta_2)+\cdot 059(\delta_1-2\delta_3-\delta_5)+q_1.
\end{aligned}$$

5.

Assuming that the probable errors in the observed quantities are all equal, and that (e) represents the error in a pair of observations combined (corresponding to the quantities (d) and (s)), then the probable error of all the p, q coefficients calculated by the formulæ (2) will be $\frac{1}{12}\sqrt{(6)}\,.\,e = \cdot 204\,e$.

The probable errors, calculated in the manner now proposed from equations (3), will be,—

Probable error of p_1 or $q_1 = \cdot 06583\sqrt{(12)}\,.\,e = \cdot 238\,e$,

„ p_2 or $q_2 = \cdot 06699\sqrt{(10)}\,.\,e = \cdot 212\,e$,

„ p_3 or $q_3 = \cdot 06904\sqrt{(9)}\,.\,e = \cdot 207\,e$,

„ $p_4 = \frac{1}{16}\sqrt{(12)}\,.\,e = \cdot 216\,e$,

„ $q_4 = \cdot 07217\sqrt{(8)}\,.\,e = \cdot 204\,e$.

The results obtained by the two methods of calculation will therefore have no sensible difference of accuracy, and as the method now proposed is believed to be both simpler and less liable to arithmetical error it may without objection be preferred. The preliminary computations on both systems, consisting of combinations of the observed quantities by addition and subtraction, are identical up to a certain point, but the formulæ (2) involve more frequent use of tables, and greater chance of error in algebraical signs, in the final operations. It may be added that much additional labour is often needlessly created by employing the hourly differences from the mean value, instead of the hourly values themselves, which are obviously sufficient for the computation of the coefficients.

The probable errors of the values of the coefficients obtained from the equations (1), will be sensibly larger than those above stated, and on the same assumptions will be as follows:—

$$\begin{aligned}
&\text{For } p_1 \text{ and } q_1 = \tfrac{1}{3}\surd(3)\,.\,e = \cdot577e,\\
&\text{,,}\ \ p_2 \text{ and } q_2 = \tfrac{1}{4}\surd(2)\,.\,e = \cdot371e,\\
&\text{,,}\ \ p_3 \text{ and } q_3 = \tfrac{1}{6}\surd(3)\,.\,e = \cdot289e,\\
&\text{,,}\ \ p_4 \qquad\quad = \tfrac{1}{4}\,.\,e \qquad = \cdot250e,\\
&\text{,,}\ \ q_4 \qquad\quad = \tfrac{1}{7}\surd(2)\,.\,e = \cdot202e.
\end{aligned}$$

The probable error of a pair of observations being rather less than $\frac{3}{4}$ths of that of a single observation, the greatest possible error of the coefficients thus found will only be about $\frac{4}{10}$ths of the probable error of one of the original observations, and when great precision is not aimed at the results thus obtained may suffice, and will not be found to differ materially from those got by the more tedious methods of calculation.

6.

If the original expression for the value of a_n is transformed into the series

$$a_n = p_0 + \mathrm{P}_1(\sin nz + \mathrm{T}_1) + \mathrm{P}_2(\sin 2nz + \mathrm{T}_2) + \&c.,$$

it follows that $\mathrm{P}_1 = \surd(p_1^2 + q_1^2)$; $\tan \mathrm{T}_1 = p_1/q_1$, and so with all the other terms of the series.

The most convenient method of computing the values of P_1 and T_1 is as follows:—

$$\log \tan \mathrm{T} = \log p - \log q;$$

from this log sin T may at once be obtained, and

$$\log \mathrm{P} = p - \log \sin \mathrm{T}.$$

The quadrant to which the angle T belongs will depend on the algebraical signs of the coefficients p q, and the following table (in which t is the angle corresponding to $+p$, $+q$) shows the cases that may arise. In it are also indicated the positions of the earliest maxima, μ_1, of the several harmonic components.

Coefficients.		Value and position of T.	Position of earliest maximum of components.			
			First order.	Second order.	Third order.	Fourth order.
$+p$	$+q$	$T = t$ 0° to 90°	$\mu_1 = 90^\circ - t$ 0° to 90°	$\mu_2 = 45^\circ - \tfrac{1}{2}t$ 0° to 45°	$\mu_3 = 30^\circ - \tfrac{1}{3}t$ 0° to 30°	$\mu_4 = 22\tfrac{1}{2}^\circ - \tfrac{1}{4}t$ 0° to $22\tfrac{1}{2}^\circ$
$-p$	$+q$	$T = 360 - t$ 270° to 360°	$\mu_1 = 90^\circ + t$ 90° to 180°	$\mu_2 = 45^\circ + \tfrac{1}{2}t$ 45° to 90°	$\mu_3 = 30^\circ + \tfrac{1}{3}t$ 30° to 60°	$\mu_4 = 22\tfrac{1}{2}^\circ + \tfrac{1}{4}t$ $22\tfrac{1}{2}^\circ$ to 45°
$-p$	$-q$	$T = 180 + t$ 180° to 270°	$\mu_1 = 270^\circ - t$ 180° to 270°	$\mu_2 = 135^\circ - \tfrac{1}{2}t$ 90° to 135°	$\mu_3 = 90^\circ - \tfrac{1}{3}t$ 60° to 90°	$\mu_4 = 67\tfrac{1}{2}^\circ - \tfrac{1}{4}t$ 45° to $67\tfrac{1}{2}^\circ$
$+p$	$-q$	$T = 180 - t$ 90° to 180°	$\mu_1 = 270^\circ + t$ 270° to 360°	$\mu_2 = 135^\circ + \tfrac{1}{2}t$ 135° to 180°	$\mu_3 = 90^\circ + \tfrac{1}{3}t$ 90° to 120°	$\mu_4 = 67\tfrac{1}{2}^\circ + \tfrac{1}{4}t$ $67\tfrac{1}{2}^\circ$ to 90°

7.

The foregoing discussion assumes that the series of quantities dealt with is truly recurrent, that is to say, that the 25th observation will be exactly coincident with the 1st, or $a_0 = a_{24}$. In fact this will rarely be the case, and it becomes necessary to ascertain what effect any non-periodic change, or want of coincidence between the beginning and end of the series will have on the values of the several p q coefficients.

In the absence of any knowledge of the law which determines such a non-periodic change, it may be assumed to be uniform for the period over which the observations extend. Further it will be convenient to refer the change to the middle of the period, or to the mean value of the series with reference to which the periodic variations are being considered, so that the non-periodic deviations will be equal and affected by opposite signs, at equal intervals on either side of the middle of the period.

Hence assuming that $2c$ is the whole non-periodic change with its proper sign, so that $a_0 + 2c = a_{24}$, the correction of any observation a_n in order to eliminate the non-periodic change which affects it, will be $\frac{1}{12}(12-n)c$; and the corrections of the quantities d, θ, Σ, and ψ are all equal to $+c$, those of Δ being 0, and of δ being $+2c$.

The correction for the mean value p_0 will be $\frac{1}{24}c$. This is equivalent to calculating the mean for the 24 hours, by adding together half of the 1st and 25th observations and the whole of the 23 intermediate ones, and dividing the sum by 24. This mean will correspond to the middle of the series, noon. As commonly calculated the mean corresponds to half an hour before noon if the series is supposed to begin with midnight, and to half an hour after noon if it begins with 1 A.M. It is to be regretted that meteorologists have not yet adopted any uniform system in these respects.

The correction for the p q coefficients as determined from the equations (2) will be as follows :—

For p_0 the mean value .... $\frac{1}{24}c$,

" p_1, p_2, p_3, p_4................ $\frac{1}{12}c = \cdot 083c$,

For q_1	$+\cdot 633c$,	For q_5	$+\cdot 112c$,
" q_2	$+\cdot 311c$,	" q_6	$+\cdot 083c$,
" q_3	$+\cdot 201c$,	" q_7	$+\cdot 064c$,
" q_4	$+\cdot 144c$.	" q_8	$+\cdot 048c$.

For the values obtained from the equations (3) they will be—*

For p_0	$\frac{1}{24}c$,	For q_1 ...	$+10c \times \cdot 06583 = \cdot 658c$,
" p_1	$+\frac{4}{3}c \times \cdot 06583$,	" q_2 ...	$+\ 5c \times \cdot 06699 = \cdot 335c$,
" p_2	$+c \times \cdot 06699$,	" q_3 ...	$+\ 3c \times \cdot 06904 = \cdot 207c$,
" p_3	$+c \times \cdot 06904$,	" q_4 ...	$+\ 2c \times \cdot 07217 = \cdot 144c$.
" p_4	$+\frac{1}{16}c$.		

8.

As it may happen that the values of the initial observation and of that corresponding to the hour immediately after the end of a series of mean hourly values, are not known, an approximate value of $2c$ may be obtained either graphically or by calculation as follows :—

If the mean values for two or three hours preceding midnight, say from 21 hours to 23 hours, be laid down graphically, and to the right of them the mean values from 0 hours to 2 hours, the ordinate of 0 hours being made coincident with that of 24 hours, the curves drawn through the points thus fixed would, if prolonged to meet one another, be continuous if there were no non-periodic disturbance. If they are not thus continuous half the distance between them measured on the ordinate of 0 hours will be the quantity c.

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* As the multipliers are here the same as those which enter into the expressions (3), the corrections may most conveniently be applied to the quantities within the brackets in those expressions.

The value may be obtained by calculation, perhaps with less trouble. Assuming that the third differences of the quantities (a) will vanish, when freed from the non-periodical variation; and that the corrected series of observed values immediately before and after midnight is $a_{22}-\frac{10}{12}c$, $a_{23}-\frac{11}{12}c$, $(a_{24}-c)$ or (a_0+c), $a_1+\frac{11}{12}c$, $a_2+\frac{10}{12}c$; and

$$a_{24}=\tfrac{4}{3}(a_{23}+a_1)-\tfrac{1}{3}(a_{22}+a_2)-a_0,$$

$$a_{24}+a_0=2c=\tfrac{4}{3}(a_{23}+a_1)-\tfrac{1}{3}(a_{22}+a_2)-2a_0,$$

$$c=\tfrac{1}{3}\{2(a_{23}+a_1)-\tfrac{1}{2}(a_{22}+a_2)\}-a_0.$$

It may also be noticed that when a series of mean values for several days is being dealt with, the quantity $2c$ will be the difference between the first observed value and the value for the hour immediately following the last of the series, divided by the number of complete days in the series.

Computation for a Yearly Period.

9.

The computation of the harmonic components of a yearly series of 365 daily values, or of 73 five-day mean values, such as are now usually calculated for the principal meteorological elements, would be extremely laborious if conducted in the usual manner, and it is therefore desirable to contrive some approximate method which shall not involve excessive arithmetical operations. For meteorological purposes the following mode of procedure will, it is believed, supply what is needed, and it may possibly be employed conveniently in other cases.

When the number of the terms of a series is exactly divisible by 2, 4, 6, and 8, it will be readily seen that the new method of computation proposed in the case of a daily series of 24 hourly terms, will be applicable in its general form; all that is necessary by way of modification being the introduction of suitable changes in the combination of the terms, and the adoption of other multipliers in place of those given in the equations (3).

The series of five-day means for a year consists of 73 terms, and the above-mentioned condition is therefore not directly complied with by it. It may, however, be easily transformed by interpolation into a series of 72 terms, which will comply with the conditions, in the following manner:—

If the original series of the 73 five-day means be represented by the numbers (0), (1), (2), (3), &c., ... (72), it will be apparent that the first term (0) corresponds with noon of the 3rd day of the year, the subsequent terms following each other at five-day intervals. The new interpolated series of 72 terms should begin at 0 hours of the 1st day of the year, and will be designated by the numbers 0, 1, 2, 3, &c., ... 71. Then if T_0 and T_{-1} represent, respectively, the mean for 24 hours of the first day of the year under computation, and the last day of the preceding year, the interpolated series will be—

$$0 = \tfrac{1}{2}(T_0 + T_{-1}),$$

$$1 = \tfrac{1}{2}((0)+(1)) + \tfrac{1}{72}((1)-(0)) = (0) + \tfrac{37}{72}((1)-(0)),$$

$$2 = \tfrac{1}{2}((1)+(2)) + \tfrac{2}{72}((2)-(1)) = (1) + \tfrac{38}{72}((2)-(1)),$$

$$3 = \tfrac{1}{2}((2)+(3)) + \tfrac{3}{72}((3)-(2)) = (2) + \tfrac{39}{72}((3)-(2)),$$

. . . .

$$35 = \tfrac{1}{2}((34)+(35)) + \tfrac{35}{72}((35)-(34)) = (34) + \tfrac{71}{72}((35)-(34)),$$

$$36 = \tfrac{1}{2}((35)+(36)) + \tfrac{36}{72}((36)-(35)) = (36),$$

$$37 = (37) + \tfrac{1}{72}((38)-(37)),$$

. . . .

$$71 = (71) + \tfrac{35}{72}((72)-(71)).$$

Moreover, for a series of 72 terms the value of z in the fundamental equation will be 5°; and designating the sums of the cosines and sines, respectively, of the successive multiples of z between z and nz by [cos 45° to 85°], and [sin 45° to 85°], and applying a similar notation to the quantities before designated by the series A, B, C, D, we shall have—

$$[A_1 \text{ to } A_{17}] = p_1[\cos 5° \text{ to } 85°] - q_1[\sin 5° \text{ to } 85°] = 10{\cdot}95188(p_1 + q_1),$$

$$[A_{19} \text{ to } A_{35}] = 10{\cdot}95188(-p_1 + q_1),$$

$$[B_1 \text{ to } B_8] = p_2[\cos 10° \text{ to } 80°] - q_2[\sin 10° \text{ to } 80°] = 5{\cdot}21503(p_2 + q_2),$$

$$[B_{10} \text{ to } B_{17}] = 5{\cdot}21503(-p_2 + q_2),$$

$$[C_1 \text{ to } C_5] = p_3[\cos 15° \text{ to } 75°] - q_3[\sin 15° \text{ to } 75°] = 3{\cdot}29788(p_3 + q_3),$$

$$[C_7 \text{ to } C_{11}] = 3{\cdot}29788(-p_3 + q_3),$$

$$[D_0 \text{ to } D_4] - [D_5 \text{ to } D_8] = p_4\{1 + 2[\cos 20° \text{ to } 80°]\} = 5{\cdot}75877p_4,$$

$$[D_1 \text{ to } D_8] = 2q_4[\sin 20° \text{ to } 80°] = 5{\cdot}67128q_4.$$

Substituting for the quantities A, B, C, D, their values in terms of the series of 73 five-day means, we obtain after reduction the values of the p q coefficients. In the reduced expressions the following symbols are employed :—

$[(0) \text{ to } (5)]+[(67) \text{ to } (72)] = \Sigma_1$ $[(0) \text{ to } (5)]-[(67) \text{ to } (72)]=\Delta_1.$

$[(6) \text{ to } (8)]+[(64) \text{ to } (66)] = \Sigma_2$ Corresponding difference $=\Delta_2,$

$[(9) \text{ to } (11)]+[(61) \text{ to } (63)] = \Sigma_3$ " " $=\Delta_3,$

$[(12) \text{ to } (17)]+[(55) \text{ to } (60)] = \Sigma_4$ " " $=\Delta_4,$

$[(18) \text{ to } (23)]+[(49) \text{ to } (54)] = \Sigma_5$ " " $=\Delta_5,$

$[(24) \text{ to } (26)]+[(46) \text{ to } (48)] = \Sigma_6$ " " $=\Delta_6,$

$[(27) \text{ to } (29)]+[(43) \text{ to } (45)] = \Sigma_7$ " " $=\Delta_7,$

$[(30) \text{ to } (35)]+[(36) \text{ to } (42)] = \Sigma_8$ " " $=\Delta_8.$

$[(0) \text{ to } (3)]+[(69) \text{ to } (72)] = \Sigma_9,$

$[(5) \text{ to } (12)]+[(60) \text{ to } (67)] = \Sigma_{10},$

$[(14) \text{ to } (21)]+[(51) \text{ to } (58)] = \Sigma_{11},$

$[(23) \text{ to } (30)]+[(42) \text{ to } (49)] = \Sigma_{12},$

$[(32) \text{ to } (35)]+[(37) \text{ to } (40)] = \Sigma_{13}.$

$$\sigma_n = (n)+(72-n). \qquad \delta_n = (n)-(72-n).$$

And let $\left(\tfrac{1}{2}\,(T_0+T_{-1})-\tfrac{1}{2}((0)+(72))\right)= K,$

then

$$45{\cdot}80753p_1 = (1-\tfrac{1}{72})\{[\Sigma_1 \text{ to } \Sigma_4]-[\Sigma_5 \text{ to } \Sigma_8]\}+\tfrac{1}{4}(-\sigma_{17}+3\sigma_{18}) + \tfrac{1}{3}\alpha+K-(36),$$

$$45{\cdot}80753q_1 = (1-\tfrac{1}{72})\{[\Delta_1 \text{ to } \Delta_8]\}-\tfrac{1}{2}\delta_0-\tfrac{1}{3}\beta,$$

$$45{\cdot}72071p_2 = (1-\tfrac{1}{72})\{[\Sigma_1\Sigma_2]-[\Sigma_3 \text{ to } \Sigma_6]+[\Sigma_7\Sigma_8]\} +\tfrac{1}{8}\{\sigma_{26}-3\sigma_8+5\sigma_9-7\sigma_{27}\}+K+(36),$$

$$45{\cdot}72071q_2 = (1-\tfrac{1}{72})\{[\Delta_1 \text{ to } \Delta_4]-[\Delta_5 \text{ to } \Delta_8]\} +\tfrac{1}{4}\{-\delta_{17}-2\delta_0+3\delta_{18}\},$$

$$\alpha=45{\cdot}57452p_3 = (1-\tfrac{1}{72})\{\Sigma_1-[\Sigma_2 \text{ to } \Sigma_4]+[\Sigma_5 \text{ to } \Sigma_7]-\Sigma_8\} +\tfrac{1}{12}\{-\sigma_{29}+3\sigma_{17}-5\sigma_5+7\sigma_6-9\sigma_{18}+11\sigma_{30}\}+K-(36),$$

$$\beta=45{\cdot}57452q_3 = (1-\tfrac{1}{72})\{[\Delta_1 \text{ to } \Delta_3]-[\Delta_4\Delta_5]+[\Delta_6 \text{ to } \Delta_8]\} +\tfrac{1}{6}\{\delta_{23}-2\delta_{11}-3\delta_0+4\delta_{12}-5\delta_{24}\},$$

$$46{\cdot}07016p_4 = (1-\tfrac{1}{72})\{\Sigma_9-\Sigma_{10}+\Sigma_{11}-\Sigma_{12}+\Sigma_{13}\} \\ +\tfrac{1}{8}\{\sigma_4-3\sigma_{13}+5\sigma_{22}-7\sigma_{31}\}+\mathrm{K}+(36),$$

$$45{\cdot}37025q_4 = (1-\tfrac{1}{72})\{[\Sigma_1\Sigma_2]-[\Sigma_3\Sigma_4]+[\Sigma_5\Sigma_6]-[\Sigma_7\Sigma_8]\} \\ +\tfrac{1}{8}\{-\delta_{26}+2\delta_{17}-3\delta_8-4\delta_0+5\delta_9-6\delta_{18}+7\delta_{27}\}.$$

The reciprocals of the numerical coefficients of the several p's and q's in the above equations will be the multipliers of the quantities represented by the right-hand portions of those equations, which will be obtained by suitable combinations of the five-day means. The multipliers are—

For p_1 and q_1 .... $\cdot 02183 = m$,

" p_2 and q_2 .... $\cdot 02187 = m(1+\cdot 002)$,

" p_3 and q_3 .... $\cdot 02194 = m(1+\cdot 005)$,

" p_4 .... $\cdot 02173 = m(1-\cdot 005)$,

" q_4 .... $\cdot 02204 = m(1+\cdot 01)$.

A table calculated to give multiples of the quantity ·02183 will evidently suffice for the other coefficients, with the small corrections shown above.

To provide for cases (which are likely to be frequent) in which the series of terms is not truly recurrent, corrections similar to those required for the 24-term series must be applied. As before, representing by T_0 and T_{-1} respectively the mean values for the first day of the period under computation, and the last day of the preceding year, and by t_0 and t_{-1} respectively the mean values for the first day of the following year and the last day of the year under computation; the quantity that must be added to the first term of the series of 72 terms, corresponding to the initial midnight of the year, to make it equal to the first term of the next yearly series, will be $2c = \frac{1}{2}(t_0+t_{-1})-\frac{1}{2}(\mathrm{T}_0+\mathrm{T}_{-1})$, and it will be found that the corrections for the several p q coefficients will be as follows:—

For p_1 .... $\frac{4}{3}c\times m = \cdot 029c$,	For q_1 .... $31\frac{1}{3}c\times m = \cdot 684c$,
" p_2 .... $c\times m(1+\cdot 002)$,	" q_2 .... $17\ c\times m(1+\cdot 002) = \cdot 372c$,
" p_3 .... $c\times m(1+\cdot 005)$,	" q_3 .... $11\ c\times m(1+\cdot 005) = \cdot 241c$,
" p_4 .... $c\times m(1-\cdot 005)$.	" q_4 .... $8\ c\times m(1+\cdot 01) = \cdot 176c$.

For p_0, or the mean of the whole series, ... $= \frac{1}{72}c$.

A similar method of computation might be adopted in the case of a yearly period of 365 days, which could be transformed by interpolation into a series of 360 days, each term of which would correspond to $\frac{365}{360}$ days $= 1 + \frac{1}{72}$ days. But the irregularities of a series of daily quantities, in meteorological discussions, would be so great, even when dealing with the mean of many years, that no practical advantage would be obtained by employing such a series; the computations would be more troublesome, and the five-day means will be preferable.

10.

The annexed Forms are proposed as supplying methods of computation from the formulæ contained in the foregoing discussion, which shall involve the least practicable quantity of arithmetical operations.

Form 1 differs little from what is believed to be the ordinary method at present. It requires tables of multiples of the sines of 15°, 45°, 60° and 75°.

Form 2 is for the proposed new method of computation. It requires tables of multiples of the special multipliers applicable to the several orders of harmonic coefficients.

Form 3 requires no comment except to mention that the value of the angles which correspond to the hours of maximum μ will facilitate the graphic representation of the several components, and appear to characterise them better than the angles T.

Forms 4 and 5 call for no special remark. But they indicate the degree of divergence that exists between the approximate values of coefficients obtained from them, and those got by the more exact methods. The figures employed are the mean hourly temperatures for one year for the month of June, at Greenwich. If the mean values for a series of years had been dealt with, the results would not have differed one with another by so much as one-tenth of a degree Fahrenheit.

Form 6 shows the method of computation from 73 five-day mean values. The figures employed are mean temperatures at Königsberg for 24 years (taken from Bessel's paper, the translation of which is given in the 'Quarterly Weather Report' of the Meteorological Office, Part IV, 1870, page [29]), which were selected in order to test the agreement of the results with those given by Bessel, calculated by the method of least squares. The values of the coefficients by the two methods are virtually identical with the exception of p_3, in which I feel satisfied that some error has been committed in Bessel's calculation; to verify this, however, would involve a very tedious computation which I have not thought it worth while to undertake. It should be observed that Bessel's coefficients, being calculated with reference to a period commencing with the first of the five-day means, that is noon on the 3rd day of the year, have to be modified to adjust them to a period commencing with the initial midnight of the year. In order to render the form complete, as a type of the proposed system of computation, values for the first and last days of the year have been interpolated, and a small non-periodical correction has been assumed to be required; but these have no practical effect on the numerical results.

For this computation a table of multiples of the multiplier (m) is required, and to save trouble a table of multiples of $\frac{1}{72}$ has also been drawn out.

NOTE.—The Tables referred to will be printed by the Meteorological Office.

Form 1.—Computation of Harmonic Coefficients of the First Four Orders, in the Form $p \cos z$, $q \sin z$, according to the method of Least Squares.

Hour.	a. 0 to 11.	a. 12 to 23.	d.	−d 4 to 7.	+d 8 to 11.	θ.	θ_0 $\theta_1-\theta_3$.	+25 +33	× 1 sin 45°	+ 25 + 23		S.	Σ.	Σ_0 1 −5	+34 +40	× 1 sin 60	+34 +35	p_2.
0–12	58·8	72·5	−137	+146	+ 16	+25			÷ 12	+ 48		1313	+ 34	2 −4	+23	sin 30	+12	
1–13	58·1	72·8	147	+127	+ 65	+45			p_3	+0·40		1309	+ 21			÷ 12	+81	+0·68
2–14	57·8	73·0	152	+ 95	+ 97	+40	θ_2	+40	× 1	+ 40		1308	+ 16	Σ_3	+ 7	× 1	+ 7	
3–15	57·5	72·9	154	+ 42	+124	+12	θ_1+_3.	+57	sin 45°	+ 40		1304	+ 7	1 +5	+ 2	sin 30	+ 1	
4–16	57·3	71·9	146						÷ 12	+ 80		1292	− 7	2 +4	+ 9	sin 60	+ 8	q_2.
5–17	57·7	70·4	127						q_3	+0·67		1281	− 19			÷ 12	+16	+0·13
				p_1.				q_2.					σ.	ψ.				
6–18	59·2	68·7	− 95	d_0.	−137	×1	− 137	d_6	− 95	×1	− 95	1279	2592	− 9	ψ_0		− 9	p_4.
7–19	62·3	66·5	− 42	d_5-d_7	− 85	sin 15°	− 22	d_5+d_7	−169	sin 75°	− 163	1288	2597	+ 6	$\frac{1}{2}(\psi_1-\psi_2)$	−13	− 7	
8–20	65·4	63·8	+ 16	4− 8	−162	sin 30°	− 81	4+ 8	−130	sin 60°	− 113	1292	2600	+19		÷12	−16	−0·13
9–21	68·1	61·6	+ 65	3− 9	−219	sin 45°	− 155	3+ 9	− 89	sin 45°	− 63	1297	2601		$\psi_1+\psi_2$	+25	+22	q_4.
10–22	69·8	60·1	+ 97	2−10	−249	sin 60°	− 216	2+10	− 55	sin 30°	− 28	1299	2591		sin 60°			
11–23	71·2	58·8	+124	1−11	−271	sin 75°	− 262	1+11	− 23	sin 15°	− 6	1300	2581			÷12		+0·18
$-a_0$.	58·8	2c				÷12	− 873			÷ 12	− 468	Sum	15562	Non-periodic correction $c = -·30$.				
$+a_{2n}$.	58·2	−0·6				p_1	−7·28			q_1	−3·90	Mean	64·84					

Coefficients	p_1.	q_1.	p_2.	q_2.	p_3.	q_3.	p_4.	q_4.	Mean.		p.	q_1.	q_2.	q_3.	q_4.	Mean.
Uncorrected.......	−7·28	−3·90	+0·68	+0·13	+0·40	+0·67	−0·13	+0·18	64·84		$\frac{1}{12}c$	·63c	·31c	·20c	·14c	$\frac{1}{24}c$
Correction	−0·03	− ·19	− ·03	− ·09	− ·03	− ·06	−0·03	− ·04	− ·01		− ·03	− ·19	− ·09	− ·06	− ·04	− ·01
Corrected.........	−7·31	−4·09	+0·65	+0·04	+0·37	+0·61	−0·16	+0·14	64·83							

Form 2.—Computation of Harmonic Coefficients of the First Four Orders, in the Form $p \cos z$, $q \sin z$, according to the proposed method.

Hour.	a.		d.	p_1.		p_3.		q_1.		q_3.		s.	Σ.		$m+\Sigma_0$	$+71$	
	0 to 11.	12 to 23.													$-n$	$+26$	
0–12	58·8	72·5	−137	d_0	−137							1313	+34		c	−3	p_2
1–13	58·1	72·8	−147	d_1-d_{11}	−271	−408	+	d_1+d_{11}	−23			1309	+21	m +37	*	+94	+0·63
2–14	57·8	73·0	−152	d_2-d_{10}	−249			d_2+d_{10}	−55			1308	+16		$m+\Sigma_3$	+44	
3–15	57·5	72·9	−154	d_3-d_9	−219			d_3+d_9	−89	−167	+	1304	+7		$+n$	−26	
4–16	57·3	71·9	−146	d_4-d_8	−162			d_4+d_8	−130			1292	−7	n −26	$5c$	−15	q_2
5–17	57·7	70·4	−127	d_5-d_7	−85	−466	−	d_6+d_7	−169			1281	−19		*	+3	+0·02
				Sum	−1123	+58	α	d_6	−95	−624	−		σ	ψ	ψ		
6–18	59·2	68·7	−95	$+\frac{1}{3}\alpha$	+19	−3	c	Sum	−561	+97	β	1279	2592	−9	$0+1-3$	−22	
7–19	62·3	66·5	−42	$\frac{4}{3}c$	−4	+55	*	$-\frac{1}{3}\beta$	−32	−9	$3c$	1288	2597	+6	c	−3	p_4
8–20	65·4	63·8	+16	*	−1108	+0·38		$10c$	−30	+88	*	1292	2600	+19	*	−25	−0·16
9–21	68·1	61·6	+65	p_1	−7·29	p_3		*	−623	+0·61		1297	2601		$1+3$	+25	
10–22	69·8	60·1	+97					q_1	−4·10	q_3		1299	2591		$2c$	−6	q_4
11–23	71·2	58·8	+124									1300	2581		*	+19	+0·14
Non-periodic correction	$-a_0$		58·8	$2c$								Sum	15562	$\frac{1}{24}c$			
	$+a_{2n}$		58·2	−0·6								Mean	64·84	−0·01	64·83	corrected mean	

The values of the coefficients are obtained by reference to tables prepared for the purpose from the quantities marked thus : *.

Form 3.—Computation of Harmonic Coefficients of the First Four Orders in the Form P sin $(z+\text{T})$.

	I.		II.		III.		IV.	
log p −log q	$p-$ $q-$	2·86273 2·61278	$p+$ $q+$	1·79934 0·30103	$p+$ $q+$	1·57978 1·78533	$p-$ $q+$	1·17609 1·14613
log tan t		10·24995		11·49831		9·79445		10·0299
t		60° 39′		72° 23′		31° 55′		46° 58′
log sin t		9·94033		9·97915		9·72325		9·86395
log p − log sin t		2·92240		1·82019		1·85653		1·31214
P		8·36		0·66		0·72		0·21
T		240° 39′		72° 23′		31° 55′		313° 2′
			$\frac{t}{2}$	36° 12′	$\frac{t}{3}$	10° 38′	$\frac{t}{4}$	11° 45′
μ		209° 21′		8° 48′		19° 22′		34° 15′

p.	q.	T.	μ First maximum.							
			I.		II.		III.		IV.	
+	+	t	90° −	t	45° −	$\frac{t}{2}$	30° −	$\frac{t}{3}$	$22\frac{1}{2}$° −	$\frac{t}{4}$
−	+	360° − t	90 +		45 +		30 +		$22\frac{1}{2}$ +	
−	−	180 + t	270 −		135 −		90 −		$67\frac{1}{2}$ −	
+	−	180 − t	270 +		135 +		90 +		$67\frac{1}{2}$ +	

Form 4.—Computation of Approximate Values of Harmonic Components of the First Four Orders.

Hour.	a.		d.	$-d_4$.	$+d_8$.	θ.	$\frac{d}{2}$.	$-$C. $-\frac{1}{6}\theta$.	A. $\frac{1}{2}d-\frac{1}{6}\theta$.	S.	Σ. 4B.	B.		Components computed from values of the p, q, coefficients.			
	0 to 11.	12 to 23.												A.	B.	C.	D.
0–12	58·8	72·5	−137	+146	+16	+25	−6·85	−·42	−7·27	1313	+34	+0·85		−7·31	+0·65	+0·40	−0·16
1–13	58·1	72·8	−147	+127	+65	+45	−7·35	−·75	−8·10	1309	+21	+0·52		−8·12	+0·58	+0·69	−0·07
2–14	57·8	73·0	−152	+95	+97	+40	−7·60	−·67	−8·27	1308	+16	+0·40		−8·38	+0·36	+0·61	+0·21
3–15	57·5	72·9	−154	+42	+124	+12	−7·70	−·20	−7·90	1304	+7	+0·18		−8·06	+0·04	+0·17	
4–16	57·3	71·9	−146				−7·30	+·42	−6·88	1292	−7	−0·18		−7·19	−0·30		
5–17	57·7	70·4	−127				−6·35	+·75	−5·60	1281	−19	−0·48		−5·84	−0·54		
											σ	ψ : 8D	D				
6–18	59·2	68·7	−95				−4·75	+·67	−4·08	1279	2592	−9	−0·11	−4·09			
7–19	62·3	66·5	−42				−2·10	+·20	−1·90	1288	2597	+6	+0·08	−2·06			
8–20	65·4	63·8	+16				+0·80	−·42	+0·38	1292	2600	+19	+0·24	+0·11			
9–21	68·1	61·6	+65				+3·25	−·75	+2·50	1297	2601			+2·28			
10–22	69·8	60·1	+97				+4·85	−·67	+4·18	1299	2591			+4·28			
11–23	71·2	58·8	+124				+6·20	−·20	+6·00	1300	2581			+6·00			
										Sum	15562						
										Mean	62·84						

Form 6.—Computation of the Harmonic Coefficients

Part II (add to Part I).

Combined multiples of differences.

Term.	q_1.		q_2.		q_3.		q_4.	
0	−1	+0·41	−2	+ 0·82	−3	+ 1·23	−4	+ 1·64
8							−3	+13·23
9							+5	−18·45
11					−2	+ 9·08		
12					+4	−16·24		
17			−1	+ 6·09			+2	−12·18
18			+3	−16·59			−6	+33·18
23					+1	− 4·50		
24					−5	+22·90		
26							−1	+ 4·02
27							+7	−27·86
Sum	+ −		+ −	6·91 16·59	+ −	33·21 20·74	+ −	52·07 58·49
÷	2	+0·41	4	− 9·68	6	+12·47	8	− 6·42
		+0·21		− 2·42		+ 2·08		− 0·80

Combined multiples of sums of terms.

Term.	p_1.		p_2.		p_3.		p_4.	
4							+1	− 4·57
5					−5	+ 17·25		
6					+7	− 13·79		
8			−3	+ 6·03				
9			+5	− 3·65				
13							−3	− 10·29
17	−1	− 7·69			+3	+ 23·07		
18	+3	+27·09			−9	− 81·27		
22							+5	+ 79·15
26			+1	+ 21·10				
27			−7	−161·42				
29					− 1	− 24·08		
30					+11	+268·95		
31							−7	−179·97
Sum	+ −		+ −	27·13 165·07	+ −	309·27 119·14	+ −	79·15 194·83
÷	4	+19·40	8	−137·94	12	+190·13	8	−115·68
(36)	−	+ 4·85 12·73	+	− 17·24 12·73	−	+ 15·84 12·73	+	− 14·46 12·73
$x-z$		− 7·88 − 0·17		− 4·51 − 0·17		+ 3·11 − 0·17		− 1·73 − 0·17
		− 8·05		− 4·68		+ 2·94		− 1·90

Pairs of terms. — Five-day means.

Sums. σ.	Differences. δ.	Term.	1st half-year.	2nd half-year.	Ter
− 7·67	−0·41	0	− 4·04	− 3·63	72
*	*	1	4·12	2·34	71
*	*	2	3·72	3·09	70
*	*	3	2·45	1·79	69
− 4·57	*	4	3·36	− 1·21	68
− 3·45	*	5	2·61	− 0·84	67
− 1·97	*	6	2·17	+ 0·20	66
*	*	7	3·09	+ 0·82	65
− 2·01	−4·41	8	3·21	1·20	64
− 0·73	−3·69	9	2·21	1·48	63
*	*	10	1·74	2·90	62
*	−4·54	11	1·26	3·28	61
*	−4·06	12	− 0·72	3·34	60
+ 3·43	*	13	− 1·12	4·55	59
*	*	14	− 0·11	5·47	58
*	*	15	+ 0·05	5·50	57
*	*	16	− 0·25	5·75	56
+ 7·69	−6·09	17	+ 0·80	6·89	55
+ 9·03	−5·53	18	+ 1·75	7·28	54
*	*	19	3·02	8·48	53
*	*	20	3·95	8·82	52
*	*	21	4·96	9·46	51
+15·83	*	22	5·37	10·46	50
*	−4·50	23	6·12	10·62	49
*	−4·58	24	7·40	11·98	48
*	*	25	7·93	12·42	47
+21·10	−4·02	26	8·54	12·56	46
+23·06	−3·98	27	9·54	13·52	45
*	*	28	9·75	13·76	44
+24·08	*	29	10·07	14·01	43
+24·45	*	30	10·33	14·12	42
+25·71	*	31	11·50	14·21	41
*	*	32	11·41	14·04	40
*	*	33	11·02	13·27	39
*	*	34	11·45	13·64	38
*	*	35	+12·05	+13·63	37
		36	+12·73		

Mean, 31st Dec. „ 1st Jan.	T_{-1} T_0	−4·00 −4·02	−8·02
Mean, 31st Dec. „ 1st Jan.	t_{-1} t_0	−3·74 −3·80	−7·54
$\frac{1}{2}(T_{-1}+T_0)$ $\frac{1}{2}(t_{-1}+t_0)$ $\frac{1}{2}(0+72)$	x y z	−4·01 −3·77 −3·84	
$x-z$		−0·17	
$y-x = 2c$		+0·24	Mean ·014

fficients of the First Four Orders for a Series of 73 Five-Day Means.

ans.		Partial sums of terms.		Combined half-years' sums.		Sums and differences of half-years' sums of terms.		
half-ar.	Term.	1st half year.	2nd half-year.	3 and 6 terms.	4 and 8 terms.	Sums.	Differences.	
						Σ.	Δ.	
·63	72							
·34	71							
·09	70				−1433			
·79	69	−1433	−1085		−1085	−2518	*	A
·21	68	−336	−121	−2030				
·84	67	−261	−084	−1290		−3320	−740	B
·20	66							
·82	65			−847				
·20	64	−847	+222	+222		−625	−1069	C
·48	63							
·90	62			−521				
·28	61	−521	+766	+766	−1701	+245	−1287	D
·34	60	−072	+334		+1238	−463	*	E
·55	59	−112	+455					
·47	58							
·50	57							
·75	56			−135				
·89	55	+049	+2361	+3150		+3015	−3285	F
·28	54							
·48	53							
·82	52				+1417			
·46	51	+1368	+3404		+5765	+7182	*	G
·46	50	+537	+1046	+2517				
·62	49	+612	+1062	+5512		+8029	−2995	H
·98	48							
·42	47			+2387				
·56	46	+2387	+3696	+3696		+6083	−1309	I
·52	45							
·76	44			+2936				
·01	43	+2936	+4129	+4129	+6968	+7065	−1193	K
·12	42	+1033	+1412		+10299	+17267	*	L
·21	41	+1150	+1421					
·04	40							
·27	39				+4593			
·64	38			+6776	+5458			
·63	37	+4593	+5458	+8291		+10051	*	M
						+15067	−1515	N
			+36	+1273				
−8·02			Sum +	41655				
			−	4823				
−7·54			÷73	36832				
			Mean	+5·045				

Non-periodic correction.

Mean.	p_1.	$p_{2\text{ to }4}$.	q_1.	q_2.	q_3.	q_4.
·014c	·029c	·022c	·684c	·372c	·241c	·176c

Part I.

Differences.	Combined differences of half-year's terms						
		q_1.		q_2.		q_3.	
B	+	−7·40	+	−7·40	+	−7·40	+
C	+	−10·69	+	−10·69	+	−10·69	+
D	+	−12·87	+	−12·87	+	−12·87	−
F	+	−32·85	+	−32·85	−	+32·85	−
H	+	−29·95	−	+29·95	−	+29·95	+
I	+	−13·09	−	+13·09	+	−13·09	+
K	+	−11·93	−	+11·93	+	−11·93	−
N	+	−15·15	−	+15·15	+	−15·15	−
Sum	+		+	70·12	+	62·80	+
	−		−	63·81	−	71·13	−
$-\frac{1}{72}$	−	133·93	+	6·31	−	8·33	+
		1·86		0·09		0·12	
P*t* 2	−	132·07	+	6·22	−	8·21	+
	+	0·21	−	2·42	+	2·08	−
$-\frac{\beta}{3}$	−	131·86	+	3·80	−	6·13	+
	+	2·04				β	
	−	129·82					
Tab.		−2·84		+0·08		−0·13	
Corr.		+0·08		+0·04		+0·03	
	q_1	−2·76	q_2	+0·12	q_3	−0·10	q_4

Sums.	Combined sums of half-year's terms.						
		p_1.		p_2.		p_3.	
A	*		*		*		+
B	+	−33·20	+	−33·20	+	−33·20	*
C	+	−6·25	+	−6·25	−	+6·25	*
D	+	+2·45	−	−2·45	−	−2·45	*
E	*		*		*		−
F	+	+30·15	−	−30·15	−	−30·15	*
G	*		*		*		+
H	−	−80·29	−	−80·29	+	+80·29	*
I	−	−60·83	−	−60·83	+	+60·83	*
K	−	−70·65	+	+70·65	+	+70·65	*
L	*		*		*		−
M	*		*		*		+
N	−	−150·67	+	−150·67	−	−150·67	*
Sum	+	32·60	+	221·32	+	218·02	+
	−	401·89	−	213·17	−	216·47	−
$-\frac{1}{72}$	−	369·29	+	8·15	+	1·55	−
		5·13		·11		·02	
P*t* 2	−	364·16	+	8·04	+	1·53	−
	−	8·05	−	4·68	+	2·94	−
$+\frac{a}{3}$	−	372·21		+3·36		+4·47	
	+	1·49				α	
	−	370·72					
Tab.		−8·09		+0·07		+0·10	
Corr.		0		0		0	
	p_1	−8·09	p_2	+0·07	p_3	+0·10	p_4

The quantities in the line marked "Tab." are found ar's terms.

		q_4.
·40	+	− 7·40
69	+	−10·69
87	−	+12·87
·85	−	+32·85
95	+	−29·95
09	+	−13·09
93	−	+11·93
15	−	+15·15
80	+	72·80
13	−	61·13
33	+	11·67
12		0·15
21	+	11·52
08	−	0·80
13	+	10·72
13		+0·23
03		+0·02
10	q_4	+0·25

s terms.

		p_4.
	+	− 25·18
·2c	*	
·25	*	
·45	*	
	−	+ 4·63
·15	*	
	+	+ 71·82
·29	*	
·83	*	
·65	*	
	−	−172·67
	+	+100·51
·67	*	
·02	+	176·96
·47	−	197·85
·55	−	20·89
·02		·29
·53	−	20·60
·94	−	1·90
·47		− 22·50
·10		− 0·49
		0
·10	p_4	− 0·49

re found in the

$y-x = 2c$	+0·24

Mean
·014
·00

Mean.	p_1.	$p_{2 \text{ to } 4}$.	q_1.	q_2.	q_3.	q_4.
·014c	·029c	·022c	·684c	·372c	·241c	·176c
·00	·00	·00	+ ·08	+ ·04	+ ·03	+ ·02

	p_1	−8·09	p_2	+ 0·07	p_3	+ 0·10	p_4

The quantities in the line marked "Tab." are found
tables of multiples of (m) from the figures in the line
diately above.

·10	p_4	— 0·49

re found in the
the line imme-

Form 5.—Computation of Approximate Values of the Harmonic Coefficients of the First Three Orders.

Hours.	Observed.		Differences.	Sums.	p.			q.		
	0 to 11.	12 to 23.								
0........	58·8	72·5	−137	1313	$\frac{1}{2}d_0$	−6·9		$\frac{1}{2}d_6$	−4·8	
1........	58·1	72·8	*	*	$-p_3$	−0·4		$+q_3$	+0·7	
2........	57·8	73·0	−152	*		−7·3	p_1		−4·1	q_1
3........	57·5	72·9	*	1304	S_0	1313		S_3	1304	
4........	57·3	71·9	−146	*	$-S_6$	1279		$-S_9$	1297	
5........	57·7	70·4	*	*	÷4	+34		÷4	+ 7	
6........	59·2	68·7	− 95	1279		+0·8	p_2		+0·2	q_2
7........	62·3	66·5	*	*	d_0	−137		d_2	−152	
8........	65·4	63·8	+ 16	*	$-d_4$	+146		$-d_6$	+ 95	
9........	68·1	61·6	*	1297	$+d_8$	+ 16		$+d_{10}$	+ 97	
10........	69·8	60·1	+ 97	*	÷6	+ 25		÷6	+ 40	
11........	71·2	58·8	*	*		+0·4	p_3		+0·7	q_3

Form 6.—Compntation of the Harmonic Coefficients of the First Fonr Orders for a Series of 73 Five-Day Means.

Part II (add to Part I).

Combined multiples of differences.

Term.	q_1.		q_2.		q_3.		q_4.	
0	−1	+0·41	−2	+ 0·82	−3	+ 1·23	−4	+ 1·64
8							−3	+13·23
9							+5	−18·45
11					−2	+ 9·08		
12					+4	−16·24		
17			−1	+ 6·09			+2	−12·18
18			+3	−16·59			−6	+33·18
23					+1	− 4·50		
24					−5	+22·90		
26							−1	+ 4·02
27							+7	−27·86
Sum	+ −		+ −	6·91 16·59	+ −	33·21 20·74	+ −	52·07 58·49
÷	2	+0·41	4	− 9·68	6	+12·47	8	− 6·42
		+0·21		− 2·42		+ 2·08		− 0·80

Combined multiples of sums of terms.

Term.	p_1.		p_2.		p_3.		p_4.	
4							+1	− 4·57
5					−5	+ 17·25		
6					+7	− 13·79		
8			−3	+ 6·03				
9			+5	− 3·65				
13							−3	− 10·29
17	−1	− 7·69			+3	+ 23·07		
18	+3	+27·09			−9	− 81·27		
22							+5	+ 79·15
26			+1	+ 21·10				
27			−7	−161·42				
29					− 1	− 24·08		
30					+11	+268·95		
31							−7	−179·97
Sum	+ −		+ −	27·13 165·07	+ −	309·27 119·14	+ −	79·15 194·83
÷	4	+19·40	8	−137·94	12	+190·13	8	−115·68
(36)	−	+ 4·85 12·73	+	− 17·24 12·73	−	+ 15·84 12·73	+	− 14·46 12·73
$x-z$		− 7·88 − 0·17		− 4·51 − 0·17		+ 3·11 − 0·17		− 1·73 − 0·17
		− 8·05		− 4·68		+ 2·94		− 1·90

Pairs of terms.		Five-day means.				Partial sums of terms.		Combined half-years' sums.		Sums and differences of half-years' sums of terms.		
Sums.	Differences.	Term.	1st half-year.	2nd half-year.	Term.	1st half year.	2nd half-year.	3 and 6 terms.	4 and 8 terms.	Sums.	Differences.	
σ.	δ.									Σ.	Δ.	
− 7·67	−0·41	0	− 4·04	− 3·63	72							
*	*	1	4·12	2·34	71							
*	*	2	3·72	3·09	70				−1433			
*	*	3	2·45	1·79	69	−1433	−1085		−1085	− 2518	*	A
− 4·57	*	4	3·36	− 1·21	68	− 336	− 121	−2030				
− 3·45	*	5	2·61	− 0·84	67	− 261	− 084	−1290		− 3320	− 740	B
− 1·97	*	6	2·17	+ 0·20	66							
*	*	7	3·09	+ 0·82	65			− 847				
− 2·01	−4·41	8	3·21	1·20	64	− 847	+ 222	+ 222		− 625	−1069	C
− 0·73	−3·69	9	2·21	1·48	63							
*	*	10	1·74	2·90	62			− 521				
*	−4·54	11	1·26	3·28	61	− 521	+ 766	+ 766	−1701	+ 245	−1287	D
*	−4·06	12	− 0·72	3·34	60	− 072	+ 334		+1238	− 463	*	E
+ 3·43	*	13	− 1·12	4·55	59	− 112	+ 455					
*	*	14	− 0·11	5·47	58							
*	*	15	+ 0·05	5·50	57							
*	*	16	− 0·25	5·75	56			− 135				
+ 7·69	−6·09	17	+ 0·80	6·89	55	+ 049	+2361	+3150		+3015	−3285	F
+ 9·03	−5·53	18	+ 1·75	7·28	54							
*	*	19	3·02	8·48	53							
*	*	20	3·95	8·82	52				+1417			
*	*	21	4·96	9·46	51	+1368	+3404		+5765	+7182	*	G
+15·83	*	22	5·37	10·46	50	+ 537	+1046	+2517				
*	−4·50	23	6·12	10·62	49	+ 612	+1062	+5512		+ 8029	−2995	H
*	−4·58	24	7·40	11·98	48							
*	*	25	7·93	12·42	47			+2387				
+21·10	−4·02	26	8·54	12·56	46	+2387	+3696	+3696		+ 6083	−1309	I
+23·06	−3·98	27	9·54	13·52	45							
*	*	28	9·75	13·76	44			+2936				
+24·08	*	29	10·07	14·01	43	+2936	+4129	+4129	+6968	+ 7065	−1193	K
+24·45	*	30	10·33	14·12	42	+1033	+1412		+10299	+17267	*	L
+25·71	*	31	11·50	14·21	41	+1150	+1421					
*	*	32	11·41	14·04	40							
*	*	33	11·02	13·27	39							
*	*	34	11·45	13·64	38			+6776	+4593			
*	*	35	+12·05	+13·63	37	+4593	+5458	+8291	+5458	+10051	*	M
		36	+12·73							+15067	−1515	N
							+36	+1273				
							Sum + −	41655 4823				
							÷73	36832				
							Mean	+5·045				

Mean, 31st Dec. „ 1st Jan.	T_{-1} T_0	−4·00 −4·02	−8·02
Mean, 31st Dec. „ 1st Jan.	t_{-1} t_0	−3·74 −3·80	−7·54
$\frac{1}{2}(T_{-1}+T_0)$ $\frac{1}{2}(t_{-1}+t_0)$ $\frac{1}{2}(0+72)$	x y z	−4·01 −3·77 −3·84	
$x-z$		−0·17	
$y-x=2c$		+0·24	

Non-periodic correction.

Mean.	p_1.	$p_{2\text{ to }4}$.	q_1.	q_2.	q_3.	q_4.
·014c	·029c	·022c	·684c	·372c	·241c	·176c
·00	·00	·00	+ ·08	+ ·04	+ ·03	+ ·02

Part I.

Combined differences of half-year's terms.

Differences.	q_1.		q_2.		q_3.		q_4.	
B	+	− 7·40	+	− 7·40	+	− 7·40	+	− 7·40
C	+	−10·69	+	−10·69	+	−10·69	+	−10·69
D	+	−12·87	+	−12·87	+	−12·87	−	+12·87
F	+	−32·85	+	−32·85	−	+32·85	−	+32·85
H	+	−29·95	−	+29·95	−	+29·95	+	−29·95
I	+	−13·09	−	+13·09	+	−13·09	+	−13·09
K	+	−11·93	−	+11·93	+	−11·93	−	+11·93
N	+	−15·15	−	+15·15	+	−15·15	−	+15·15
Sum	+ −		+ −	70·12 63·81	+ −	62·80 71·13	+ −	72·80 61·13
$-\frac{1}{72}$	−	133·93 1·86	+	6·31 0·09	−	8·33 0·12	+	11·67 0·15
Pt 2	− +	132·07 0·21	+ −	6·22 2·42	− +	8·21 2·08	+ −	11·52 0·80
$-\frac{\beta}{3}$	− +	131·86 2·04	+	3·80	−	6·13 β	+	10·72
	−	129·82						
Tab. Corr.		−2·84 +0·08		+0·08 +0·04		−0·13 +0·03		+0·23 +0·02
	q_1	−2·76	q_2	+0·12	q_3	−0·10	q_4	+0·25

Combined sums of balf-year's terms.

Sums.	p_1.		p_2.		p_3.		p_4.	
A	*		*		*		+	− 25·18
B	+	− 33·20	+	− 33·20	+	− 33·20	*	
C	+	− 6·25	+	− 6·25	−	+ 6·25	*	
D	+	+ 2·45	−	− 2·45	−	− 2·45	*	
E	*		*		*		−	+ 4·63
F	+	+ 30·15	−	− 30·15	−	− 30·15	*	
G	*		*		*		+	+ 71·82
H	−	− 80·29	−	− 80·29	+	+ 80·29	*	
I	−	− 60·83	−	− 60·83	+	+ 60·83	*	
K	−	− 70·65	+	+ 70·65	+	+ 70·65	*	
L	*		*		*		−	−172·67
M	*		*		*		+	+100·51
N	−	−150·67	+	−150·67	−	−150·67	*	
Sum	+ −	32·60 401·89	+ −	221·32 213·17	+ −	218·02 216·47	+ −	176·96 197·85
$-\frac{1}{72}$	−	369·29 5·13	+	8·15 ·11	+	1·55 ·02	−	20·89 ·29
Pt 2	− −	364·16 8·05	+ −	8·04 4·68	+ +	1·53 2·94	− −	20·60 1·90
$+\frac{\alpha}{3}$	− + −	372·21 1·49 370·72		+ 3·36		+ 4·47 α		− 22·50
Tab.		−8·09		+ 0·07		+ 0·10		− 0·49
Corr.		0		0		0		0
	p_1	−8·09	p_2	+ 0·07	p_3	+ 0·10	p_4	− 0·49

The quantities in the line marked "Tab." are found in the tables of multiples of (m) from the figures in the line immediately above.